Democratic Access to Powerful Mathematics in a Developing Country

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13.1. INTRODUCTION

The arrival of a new millennium has transformed the expectation of educational change in contemporary societies. There is a renewed interest in education; every country, in one way or another, is preparing to face the future with education as the sustaining foundations of what has been called the century of information and knowledge. Therefore, the transformation of educational systems will have to incorporate the immense scientific and technological developments of the past decades (which are in part responsible for our expectations for the new century). Every country, depending on its sociocultural and economic conditions, is preparing to face new challenges in education, as education is being understood not only as a systematic solution to most immediate social needs, but also as a means to face the unknown and the unexpected. This suggests the need to reformulate what is taught, as well as how and why it is taught. Consideration of these issues will allow us to define new requirements of mathematics education that must be identified at all levels of learning from elementary school to university. Thus, we foresee important transformations in the field of curricular design and development, as well as in the application of new learning tools.

In countries such as Mexico, in addition to teaching specific skills, we will need to teach students to think critically about the ongoing changes in the world and about how these changes can affect educational and national realities. It will be necessary for education to generate the ability to respond with a spirit of innovation when faced with the changing reality.

In Mexico, access to knowledge cannot be regarded as a politically neutral issue because there is an obvious problem of exclusion for those who are on the margin
of the educational process at any of its levels. Our inclusion in the contemporary world of globalization demands that we have the critical ability to transfuse scientific and technological developments into our educational realities. We cannot forget that a preexisting school culture has left a significant mark on the players within the educational system. This school culture requires the gradual reorientation of its practices and cognitive and epistemological assumptions to gain access to the powerful ideas of mathematics and to the development of equally fundamental skills for these ideas, such as exploring, modeling, handling of information, and the ability to systematize. This is not always possible under the traditional teaching model that has dominated education until recently.

Today in Mexico, 7 out of 10 Mexicans live in urban areas; 25% of the population is concentrated in four major cities, and a population of 10 million is dispersed in very small communities. Great ethnic diversity and high levels of poverty and illiteracy live side by side in these communities. This simultaneous concentration and dispersion is a negative factor impacting the development of education at a national level (Academia Mexicana de Ciencias, 1999). The greatest challenge is the development of an education system that will enable us to deal with this diverse social and cultural reality. Responses from the educational community must generate curricular reorganization processes around conceptual fields that promote critical thinking and an evaluation of technological environments.

The projects, presented in this chapter show different ways of incorporating educational research into curricular development. Educational research has identified powerful ideas that can be used to develop a curriculum. Nonetheless, such research cannot take place on a single front, given the social and cultural characteristics of our country.

As a country, Mexico must deal with a twofold problem: the education of dispersed groups and that of large urban nuclei. From this stems the nature of the projects discussed in this chapter.

Our first project responds to the needs of the most vulnerable communities. In contrast, the second project, is aimed at a school population made up of regular students within the education system. It is possible and feasible to cultivate powerful ideas that generate different levels of mathematical thinking through the mediation of computing instruments. In countries such as Mexico, these new applied research projects are the key to assimilating scientific and technological knowledge.

From an international perspective, curriculum innovation projects reveal three essential commonalities (Black & Atkins, 1996): (a) the importance of the students' practical work; (b) the importance of making explicit the link between different scientific fields, as well as between these fields and other knowledge domains; and (c) acknowledgement of the fact that mathematics and science are ways of knowing and explaining the world.

13.2. BACKGROUND

Here we describe some basic characteristics of the Mexican educational system. Essential changes in basic education programs are suggested by the present curriculum reforms. In particular, the structuralist approach to curriculum has been replaced by an approach based on problem solving (Secretaria de Educación Pública, 1995). Other changes include the need to recognize that learning is not an automatic consequence of teaching.

In primary school, the problem-solving approach is oriented toward concrete situations, and students are encouraged to discuss and compare their own ideas. This goes hand in hand with the fact that mathematical knowledge is constructed through successive abstractions, and in addition, it makes clear the instrumental character of mathematics. In secondary school, it is expected that students (12- to 16-year-olds)
will strengthen their problem-solving skills and be able to transfer these skills to other domains. In other words, they are expected to begin the complex process of decontextualizing their own knowledge. We understand this process as one that enables students to establish rich connections with their knowledge.

At the secondary-school level, mathematical content is grouped into five domains: arithmetic, algebra, geometry, data handling, and probability. Geometry is regaining the place it once had in programs preceding the structuralist approach. The current approach to geometry focuses on the development of reasoning skills from hypotheses constructed and tested by the students themselves. On the other hand, algebra is studied as a modeling and problem-solving approach.

It is within this general background that we will try to situate two development projects, both of which were elaborated and implemented in Mexico with the purpose of improving and eventually transforming mathematics education. These projects are compatible with the development of the educational and cultural conditions necessary to make possible democratic access to powerful mathematics.

At this point, we begin elucidating our notion of access to powerful mathematics through school culture: It mainly has to do with providing the student with the opportunity to (a) experience the construction of mathematical knowledge within school according to her or his level of development; (b) develop her or his creativity through exploration of the different approaches that emerge from discussing questions posed within the classroom; and (c) developing her or his own computing techniques or procedures.

We can assume that students will, sooner rather than later, start to bring calculators to school (if this has not already happened). The access to powerful ideas in the context of teaching and learning with calculators can mean a change in the way we work with decimal numbers, for example, reinforcing the powerful idea of approximation. To a great extent, this is made possible by the positional meaning of digits in decimal expressions. This is undoubtedly one of the central features of the representation system. Unfortunately, the uncritical use of calculators leads to the idea that all decimal expressions are finite. If we want students to reach a higher conceptual level (which is the same as saying that they have taken on a powerful idea), we must propose teaching that goes into depth regarding the idea of approximation and removes the false idea that all numerical decimal expressions are finite. This is achieved through suitable teaching models, such as the systematic study of the change of units of measure in calculations of length, area, and volume. These activities bring out another important idea, that of “better approximation.” The synthesis of the ideas of approximation and better approximation constitute an example of a powerful idea that it is worth developing at different levels of the education system and with various technological resources (NCTM, 2000).

Other powerful ideas will come with numerical calculation. It will not be necessary to always insist on the accuracy of numerical results. The development of calculating skills combined with the numeric control of hand calculators might be an invaluable resource for students outside the classroom. In the particular case of geometry, the possibilities of dynamic tools such as Cabri might lead to a change in the current conception of school geometry as a pedagogical model, which closely follows the axiomatic organization of geometric knowledge. Nonetheless, we should not forget that large portions of the Mexican population live in dispersed communities, particularly in the country. Hence, the educational system will also need to provide answers that favor the integration and development of these communities. Our project “Dialogue and Discovery” faces these challenges. This project has minimal technological requirements (if the resource to the technology of writing can be understood that way). But the growing complexity of our world requires people whose education has trained them to develop their activities with increasingly greater levels of systematicity. Mathematical education can contribute in a significant way to
the achievement of these goals as long as teaching models allow the appropriation of conceptual tools to model and formalize situations, for example, to introduce a higher level of predictability and therefore better control over the consequences of social action.

13.3. DIALOGUE AND DISCOVERY: A CURRICULAR DEVELOPMENT PROPOSAL FOR SMALL RURAL SCHOOLS

In this section, we will deal with the question of how to introduce powerful mathematical ideas into rural schools where young instructors, graduates from 9th grade and aged between 16 and 21 years old, have to work with groups of children from 1st to 6th grade simultaneously, all in a single classroom.

Thus, we provide and discuss some of the arguments that emerged during the process of elaborating a curricular development program, the "Community Courses," aimed at schools with the characteristics mentioned above. Many of the decisions that were taken in the particular case of mathematics are part of a more general strategy that oriented the program's global design and which includes the areas of Spanish and science. Hence, we will refer to this general strategy by illuminating the specific case of mathematics.

13.3.2. The Community Courses

The Community Courses program represents an alternative to representative school organization, providing primary education to children living in very small rural communities (less than 100 people) dispersed throughout the country, where it is not cost-effective to install regular schools. The program was instituted 25 years ago by a decentralized government agency, the National Council for the Promotion of Education (CONAFE). Today, there are around 15,000 courses serving approximately 140,000 children. The program works as a contract between the community, the instructor, and the CONAFE. The community is in charge of providing a physical space to run the courses, food and shelter for instructors, and for supervising the adequate operation of the courses. The instructors are usually young people from the rural community, between 16 and 21 years old, who have graduated from secondary school (9th grade) or, less frequently, from high school (12th grade); they work only for a period of 1 or 2 years within the community, receiving a small payment from CONAFE. Once their working period is finished, CONAFE grants them a scholarship to continue with their own studies.

One instructor deals simultaneously with between 10 and 30 pupils from the six different grades at the primary level. They receive a 2-month intensive training during the summer and subsequent timely supervisions during the year. As the reader may have guessed, we are talking about the most modest schools in the country.

In 1975, CONAFE asked a group of education researchers from the Education Research Department at Cinvestav (the Center for Research and Advanced Studies) to develop both a pedagogic model and a set of supporting materials for the program. These were elaborated between 1975 and 1978 and were used for 15 years. In 1989, the revision and actualization of these materials was considered and assigned again to Cinvestav. During the next 4 years we elaborated a new set of materials, nine textbooks in total that comprise the Dialogue and Discovery, series which will be discussed here. These books have been used in the Community Courses program since 1994.

In the process of creating this series, we used contributions from two different approaches to the problem of classroom teaching: (a) research on the didactics of
specific disciplines in this case mathematics, and (b) ethnographic research on school culture and teaching practices. We are particularly interested here in showing the way in which these two approaches were integrated to answer the specific problems that the development of curricula entails.

To grasp the meaning of “powerful ideas” in mathematics learning, we begin by examining existing research on the didactics of mathematics.

13.3.3. Powerful Ideas in Mathematics Learning in Primary School

Today the greatest challenge in mathematics teaching in primary school is still the same one that the greatest curriculum reformers described almost 40 years ago: overcoming a tendency to reduce the discipline into the teaching of mechanisms that are supplied with few meanings.

Beyond the differences among contemporary approaches, there is a consensus among the research community that we must contextualize the knowledge taught in basic education to let construction of meaning of mathematical situations to take place instead of a premature formalization.

For almost twenty years, we have studied the didactic conditions that favor the learning process of mathematical knowledge in the primary school classroom. Mathematical knowledge is considered a mediating tool to deal with specific problems. Learning is considered an outcome from an interaction between the individual and the environment that is full of interference. Our efforts focus on studying the specific milieu that will favor the construction of mathematical knowledge in the classroom (Brousseau, 1987).

Exploration within the classroom is the main methodological tool we have used in our studies. The design of teaching strategies relative to a tangible idea is carried out from an analysis of the problems in which that idea works as a solving tool. Moreover, to support the development of solution strategies, the situation should allow students to see for themselves how they were able to solve a problem, or to what extent they solved it. The latter is a form of programmed verification that is destined to be replaced progressively by a form of semantic validation through argumentation (Block, 1991).

From this perspective, introducing powerful mathematical ideas into the primary school classroom gives students studying arithmetic and geometry the chance to experience personal, and therefore significant, construction of mathematical knowledge. This provides them with appropriate tools, according to their age and cognitive abilities, to perform creative work by elaborating and testing their own conjectures and constructing or refining their own calculating techniques.

13.3.4. Research and Curricular Development

Ethnographic studies about teaching practices have also contributed significantly to eradicating the myth of the apparent transparency in the relation between pedagogic models and real teaching practices. They have helped us to understand that teachers construct their practice from their own experience and, in specific school conditions, with the cultural resources at hand (Rockwell & Mercado, 1988). Thus, when designing materials for the Dialogue and Discovery project, we assumed that their effective use does not depend only on the instructor’s adaptation to the materials but primarily on how the proposal can be adapted to the instructor’s working habits. In addition, we assumed that there is a creative relationship between the instructor and the textbooks, a relation in which “textbooks are interpreted and proposals are both reformulated and selectively integrated into a practice that is constructed every day within the classroom” (Rockwell, Block, & et al., 1993). The main challenges and interests of
this project emerge precisely from the extreme conditions of austerity in which the proposal should work, paired with the purpose of developing a high-quality program that is able to recover existing knowledge on teaching and learning processes.

13.3.5. On Methodology

Before producing the materials, we visited diverse communities to learn the general conditions in which the instructors had to work, as well as to observe their working practices. In addition, we met with members of technical teams from different states in the country and learned about the main experimental difficulties experienced by the instructors, as well as about their opinions on the characteristics that the proposal should have. From this preliminary data, we made important decisions regarding the proposal's general design.

During the design stage, we systematically tested the textbooks in diverse communities throughout the country. The main purpose of these tests was to confirm that the situations were adequately designed both for the students and for the instructors (i.e., that they were sufficiently clear and feasible in terms of their organization). After the first version of the proposal was available, we conducted an evaluation in several communities during a single school year; at the end of the year, we received the materials used with numerous comments from the instructors and students.

13.3.5.1. Selection and Reorganization of Contents

The current national curriculum was employed as a framework to produce the Dialogue and Discovery project. Nevertheless, because there was a need to adapt the program to actual available time which were substantially shorter than the official standard teaching time, the contents of the four areas were submitted to a careful selection and reorganization. We knew that, at this level, children could profit from dealing with certain unit partition experiences. The teacher's expectation of obtaining precise results hinder the formative value of these experiences, however. This is why we decided to introduce fractions in the 3rd grade. In the same way, we decided to exclude certain aspects from this topic, such as fraction multiplication and division, that are not fundamental at a basic level and that usually generate significant teaching and learning difficulties.

13.3.5.2. Separating the Primary Curriculum into Three Levels Instead of Six

The organization of the children's and the instructor's work was intended both to make it possible for the instructor to simultaneously pay attention to the students of all six grades and to make the effects of the rural conditions result in a positive outcome.

First, the curriculum was separated into three levels, with two grades per level. In this way, the instructor is able to plan the tasks for three groups of children instead of six, and during the class he or she could distribute attention between the three groups. This means that the children works twice, once per year, with each level's tasks. This form of organization is supported by the understanding of learning as a cyclic process in which a single problem set can be dealt with repeatedly throughout time, each time in more depth. Hence, the intention is not to "repeat" tasks, but to deal with the same situations in a more systematic in depth way, including new concepts. We also expected that the collective work of children with diverse degrees of knowledge would be beneficial to the whole group, for those who know more and for those who know less.
This consideration implies an additional reason to privilege the design of reusable or recyclable activities during the course. A situation consisting of drawing the number of objects corresponding to a given number can be a useful in assessing or revising situation; however, it is not a useful learning situation because only those who already know the number’s representation can successfully solve it. Now let us consider the following situation: the instructor puts 10 white and 10 black cards on the table. Student A leaves the classroom while student B takes a certain number of white cards, puts them in a bag, and writes on the blackboard the amount of cards he or she has put away. Student A comes back, looks at what is written on the board, and takes the same amount of black cards. They then directly compare quantities, and if they are the same, they win.

Some of the children’s typical solutions are

- Student B draws each white card introduced in the bag, or a line for each card, on the blackboard.
- Student B writes down small numbers to represent greater quantities, for example, for a collection of five cards, 2 2 1.
- Student B tries to express with one sole number the cardinal of the collection. Occasionally Student B or A makes mistakes because he or she does not know the written numerical series very well or because mistakes in counting.

Unlike the previous problem, students who are in the process of acquiring knowledge of the initial numerical series can participate in this problem. It can inspire various solutions that respond to different levels of knowledge and allow for the systematic verification of the attempts. In a similar way, applying a definite algorithm, for example division, is something that can only be done in one way, and only if one is already familiar with the concept. In contrast, solving a problem that consists of finding how many marbles each of three friends will get if they share 24 marbles equally can be approached by distributing the marbles one by one, by using additions to compute and verify, or by using multiplication to compute and verify.

We suggested making some situations more complex by manipulating numeric variables or by introducing certain constraints, so that they can still be a challenge for those who already possess a solution for the original version. One of the hardest aspects of the design is transmitting the meaning of the activities to the instructor. Let us look at an example in which the instructor’s subtle modification changes the meaning of the activity. The original situation is concerning addition and subtraction and sets out the following problem: The instructor gives a group of students a set of 20 objects that they have to count. One student leaves the classroom while the others aggregate or take away objects from the set. The student comes back and tries to figure out what the others did to the set. The aim is to encourage the student to bring into play his or her own resources to determine if any objects were added or subtracted and, if so, how many.

For example, if seven objects were taken from the set, the student can verify that objects were taken away by counting those 13 that remain. To determine how many were subtracted, he or she can add the objects that are missing to reach 20, separating them from the others and then counting them, or continue counting from 14 to 20, counting a second time on his or her hands, in the units that are missing (15 is 1, 16 is 2, etc.). When objects are added to the set, the solution is much simpler; one need only separate 20 and count the rest. In the first instance, the children think there is no way to know with certainty what they are being asked and tend to give estimated quantities: “they took about three away”. This situation is one of the first experiences children have in which a calculation allows them to anticipate quantitative events.
In one of the observed sessions, the instructor performed the situation several times in the following way:

- Each child counted the objects before going outside the classroom.
- In all the cases, the other students added objects.
- Each time a student came back, he or she was asked to separate 20 objects from the set.
- Afterwards, he or she was asked to count the rest.
- When errors occurred, the student repeated the operation.

The task was so well controlled that it ended up being a simple counting exercise. This kind of difficulty led us make as explicit as possible the tasks’ purpose, as well as the possible answers, mistakes, and procedures that the children were likely to have made within the narrow limits of space. We will return to this point later while discussing the materials’ design.

Among the tasks we are currently proposing is that of game playing. We have developed a set of 20 games, each with four variants in degree of difficulty, but all related with one or several contents from the mathematics primary curriculum. Some of the games simply represent an opportunity to exercise algorithmic thinking through the introduction of certain variants in tasks that allow the enrichment of the students’ work. Certain type of riddles, some designed ex profeso, appear to be well suited to this purpose, for instance, building “magic squares.” In contrast, other games entail the construction of a winning strategy. Students with little knowledge can play them giving rise to autonomous decision making, formulation of hypotheses, and pragmatic verification. An example of these games is the classic “race to 20,” in which two players participate. The first player writes down a number 1 or 2; the second player adds to that number a 1 or 2, and so on. The player who reaches 20 first wins. Once the child has learned how to win (and this takes time), the goal changes. For example, the goal can be to reach 21 or 22, or the size of the steps may change (for example, one can add 1, 2, and 3). These variations help the search for more general strategies.

The fact that specific learning contents are not made explicit in these tasks will presumably help the instructor allow the children to work more freely.

13.3.5.3. Supporting Texts

We produced two handbooks for instructors containing the design of class development for the three levels, as well as a game book and a manual on “the experience of being an instructor” with multiple practical recommendations and testimonies from other instructors. We also provided the national textbook, produced by the Ministry of Education. For second level students, we produced an activity card kit, and for level three we produced four workbooks, taking into account that at this level students have better writing skills and study a considerable amount of time without the instructor’s supervision.

We will discuss only some features of the instructor’s guide, those representing a part of the target user’s adaptation process in a project of curricular development. These guides contain the curricular development proposal in detail, class by class, as requested by instructors and former instructors. They argued that programming the activities was one of the most difficult tasks to carry out. As we wrote the guides, we assumed that there should not only be a working guide for the instructors, but also a guide for them to learn more mathematics and how it can be taught.

Finally, we included at the end of each chapter the assessment guides. We considered this necessary because, as is well known, the tools used for evaluating student knowledge significantly influence what is considered relevant for teaching. The
activities that we propose are basically problem situations, written or formulated orally, with eventual support from concrete materials. We also included information about possible answers, mistakes, or solving strategies, providing several criteria to help appreciate the students' progress, as well as some suggestions in case difficulties or failures to complete the tasks are identified.

Experimental work led to considerations related to form. The number of pages assigned to the development of each topic was restricted from the start, considering the time available for the instructor to read them, as well as the real class time (for instance, 46 pages for level-one mathematics); the need to be careful about the writing style was also taken into account, for example, using brief sentences and paragraphs. In addition, it was necessary to include as many figures and photographs as possible to support written materials.

13.3.6. Additional Comments on the Dialogue and Discovery Project

The development of the Dialogue and Discovery project, was an attempt to produce not only an original proposal that would contribute to the learning of meaningful knowledge, but also a feasible proposal adapted to the needs of the particular target users. In this sense, we believe the methodology that was used to develop the project, as well as the features that shaped it, constitute a contribution to the field of curricular design. The last national-scale reform for primary school (between 1992 and 1994), made by the Mexican government, used many ideas generated from this project in a substantial way. Nonetheless, an essential part of our study is still missing: An assessment of its efficiency when compared with the entire educational system operating in regular conditions. It is appropriate to mention the results of a recent study: A preliminary report from "The First National Standards Evaluation in Primary Education" (Secretaría de Educación Pública, 1995), shows that the percentage of students from the Community Courses that successfully meet established standards in mathematics is similar to the national average. Considering the working conditions of these schools, this is a significant achievement.

As Rockwell (1994) pointed out, a sophisticated research design capable of integrating and controlling variables—such as training, supervision, and real use of the materials—is needed. The latter represents one of the most urgent tasks pending in the field of curricular development. Finally, observing the children while they play the "race to 20" game, helped to alleviate our skepticism about the possibility of introducing "powerful mathematical ideas" into the most disadvantaged regions of the country.

13.4. THE TECHNOLOGY PROJECT

The burst of new technologies in education has frequently produced uncritical optimism about the possibility of transforming the foundation of educational systems. For this reason, it became necessary to challenge paradigms supporting the belief that it is through the mechanical use of these new tools that the great majority of individuals would be able to access complex and powerful mathematical notions. The access to powerful ideas has to take into account, from the start, the mediation between the technological tools and the sociocultural environment surrounding individuals and schools. The different strategies that students, teachers, and schools establish to incorporate technology depend to a great extent on the interpretative resources developed within this environment. Although we are not going to study in depth the relationship between culture and epistemology, it is an essential issue
in education. According to Balacheff and Kaput (1996), the main impact of information technology on educational systems is epistemological and cognitive because it has contributed to the production of a new form of realism in mathematical objects. This new form of realism depends on the interpretative resources provided by the sociocultural environment. At the same time, however, the existence and use of this technology can transform the initial interpretations derived from the sociocultural environment.

Thus, technology has the power to become a sociocultural and educational agent for change but this process of change is complex.

In the proceedings of a world conference on higher education, *Higher Education in the XXI Century* (chapter 12), held in Paris in October 1998, attendees concluded that it is necessary to encourage research through the construction of networks that allow democratic access to knowledge and through the adaptation of new communication technologies to national requirements. Resources of productivity lie in the technology of knowledge generation, information processing, and symbol communication (Castells, 1996).

### 13.4.2. The Project

In Mexico, as in many other countries, the incorporation of technology into the educational system is driven by a policy of primary importance. A number of educational plans exist that incorporate technology into classrooms. One of these is our national project, “*Incorporating New Technologies into School Culture: The Teaching of Mathematics in Secondary School*” funded by the Ministry of Education and the National Council for Science and Technology in Mexico (CONACYT, Project 526338S). This project is aimed at

1. Gradually incorporating various pieces of technology into the mathematics and science curricula at the secondary school level
2. Implementing the use of technology supported by a pedagogic model that allows the construction of learning environments oriented toward the improvement of mathematical education
3. Encouraging the design and use of computing environments that can improve the traditional teaching and learning methods (i.e., with paper and pencil).

The general objectives of this program are to raise education standards, to train teachers in the use of technology, and to broaden the students’ opportunities of education.

We plan to continue the program well beyond the year 2000. Initially it covered 15 States (out of 32) including both rural and urban sites. The software selected for the project includes Cabri-Géomètre, spreadsheets, and algebraic calculators (TI-92). To achieve the aims mentioned above, we have had to investigate the impact of technology on the teaching and learning processes. This has several implications for the assessment and implementation of the program. For instance, *usability* problems can affect the student’s achievement of educational goals—the raising of educational standards or the advantages that students gain from introducing technology into the classrooms. The reported results of this project take into account the progress made concerning the global goals we set for ourselves at the start of the project. Among the proposed goals is that of exploring the effects on the cognition of the students, of the insertion of computational instruments into the teaching model. We will report on those results mainly from the perspective of geometry and algebra (see Rojano’s chapter of this book) because in the development of the project, it has become apparent that these disciplines are the most promising in the context of our work and may allow students to develop powerful ideas in the mathematical education field.
13.4.3. Some Results from Fieldwork

During the development of the project, we conducted interviews and written tests to evaluate the students mathematical learning as mediated by computing tools. These are resources to obtain information that can be used as feedback for the general management and assessment of the entire project. Following are the results obtained in interviews and written tests while working with Cabri Geometry. This work is done in secondary schools (12- to 16-year-olds). One of the problems that influenced the students' completion of tasks in Cabri was learning to draw with the mouse. When the teacher asked the participants to draw geometric figures, they tended to use the mouse as a pencil metaphor. For instance, they experienced difficulties drawing segments because they moved the mouse from left to right as if they were using a pencil. Similarly, when asked to construct a triangle they proceeded by drawing three different segments instead of selecting the triangle option from the menu bar. The figure they drew looked like a triangle but did not have its properties. For instance, students were amazed when they tried to assign an area to those triangles and the environment was not responding in this respect. As a result, the users experienced difficulties in understanding relationships between the triangles they had drawn and the ones they could draw using the corresponding Cabri menu.

We also became aware of other problems, such as the difficulty in measuring angles in the Cabri environment. Some children were not able to see an 89.98° angle as a 90° angle. Our school mathematical culture is one that still demands "exactness." This is one of the obstructions we must take into account when working with numerical domains with calculators and computers. We have to be careful with considerations of usability especially when the technology was not originally designed for the cultural context in which it is being incorporated.

The above findings becomes central in evaluating implementation outcomes. If usability problems, as culturally determined, are not adequately taken into account, the introduction of the technology into schools might fail to achieve its original goals. In other words, although technological tools might be adequately designed to meet specific educational goals, if students cannot use them because it is culturally inadequate, the implementation will fail. If assessments do not take into account such usability issues and contemplate exclusively educational indicators, such as student achievement, then these assessments will probably end up recommending inadequate prescriptions in many cases.

Other problems that influenced the user-task interaction was that the technology easily shifted from being an educational tool to being an educational goal. At the end of the sessions, we asked students what they have learned, and the majority answered, "to use the computer." The latter can be due to both usability and to the novelty of using computers. Nonetheless, this is an important issue because if the software is not being used as expected, then the initial educational goal will not be achieved.

According to Balacheff and Kaput (1996), design can aid the development of fluency between diverse mathematical representations, but it can also lead to the construction of misconceptions and misunderstandings. Therefore the interface can no longer be considered as a mere superficial layer because what is involved is not mere perception but interpretation (Balacheff & Kaput, 1996, p. 475). They provide a well-documented review of existing computational technology in mathematics and describe how differences in design can affect the student's mathematical experience.

13.4.4. An Interview: The Voice of the Students

In this section, we introduce a task aimed at examining the mediating role of the calculator in the coherent oral expression of knowledge that students constructed during geometric activities (Manouchehri et al., 1998, p. 437). Geometry software
for calculators has usability limitations because a calculator’s screen resolution and speed can complicate the construction of objects in that environment. Nevertheless, it might be useful to enhance the cognitive activity of its users. We also have explored ways in which object manipulation and dragging, help students discover the invariant properties of a geometric object. We now present some of this work in relation to the central angle theorem.

The tasks we are about to present (part of a larger set of activities for secondary school pupils), were aimed at documenting how the tools provided by the calculator mediate the students’ activity of expressing a mathematical proposition. We suggest that the students’ expressions of coherent and meaningful mathematical propositions prove that they have constructed the relevant relationships or, in other words, the structural links that constitute a figure.

Looking at Fig. 13.1, the teacher asks the students to do the following:

**Teacher:** Drag point B to the right and then to the left along the arc, but look carefully and try to answer these questions:

Does angle B change when you move point B along the arc?

1. Take point A and drag it to the left and to the right. Does angle B change?
2. Take point C and drag it to the left and to the right. Does angle B change?

Before the students started to complete the task, the instructor questioned them about their previous knowledge on the subject. Only one student knew that the angle B, in Fig. 13.1, remains constant as long as one does not move points A or C. Interestingly, the students did not know how to explain this behavior. None of the participants knew the central angle theorem or that a triangle inscribed in a semicircle is always right. The following are some of the participants’ answers taken from the task sessions:

**Felipe (F):** It looks as if the angle doesn’t change, even though point B is moving!

**Manuel (M):** Let me see, I can’t see . . . maybe . . . Felipe and Manuel are talking about Fig. 13.1, while dragging point B to the left and to the right in their calculator screen.

**Teacher (T):** Do you think that the angle will change?

They both answered yes, and this is exactly what the rest of the group was expecting.
(T) (addressing the entire group): Observe what is changing and what is not changing, and try to keep on doing the task. If you find something interesting for you don’t hesitate to tell me.

The participants worked for 20 minutes on this task. Then the teacher proposed the following construction and asked the corresponding questions: Draw the segments from A and C to the center O of the circle (in Fig. 13.1). Drag point A to the left and to the right; observe angle B as well as the angle formed by the segments that connect points A and C to the center O of the circle (central angle). Repeat the operation with point C.

Move point A or point C until they are collinear with O, the center of the circle. How is angle B changed?

Felipe and Manuel, moved point A until it was collinear with point C and with point O. Finally, they moved point B, showing the teacher what they were doing at every moment. The rest of the participants observed what Felipe and Manuel had found.

F: It looks like when the angle in the middle is 180°, angle B is 90°!
T: Why are you saying so?
M: We have already tried it, and it seems that way. Look!

After 20 more minutes, nobody could further expand the argument about the rightness of the angle B. Then the teacher proposed the students measure and label the central angle and angle B. After 15 minutes, the students called the teacher and showed him a table in a notebook, one column showing the values for angle B and the other showing the values for the central angle.

F: One column is almost twice as large as the other!
T: How can you express what you found?
F: The center angle is two times greater than the other.
T: Just like that?
F: Ah . . . Within a circle the center angle is two times greater than the other.

Two out of six teams continued to complete the tasks, but only one team (Felipe and Manuel) had made additional drawings and started labeling the angles. They argued that these additional drawings and labels were intended to “help them discover.” The rest of the teams did not know what to do and did not propose any additional drawings.

What is the aim of these practical sessions? To study how students express and construct their arguments while trying to “prove” a theorem from the exploration of the links that exist between the different elements in the figures provided. Of course, exploration and expression are possible, in enhanced ways, because of the dragging capability of the software. The central angle theorem is an attraction pole, a means to link circles, rays, radii, and tangents, to create a local organization (Moreno, 1996) of a fragment of geometric knowledge. Because two important general objectives of this project are to broaden students’ educational opportunities and, concurrently, to train teachers in the use of technology, we find it compulsory to articulate a reflection on the use of these computational tools and the environment wherein they are explored.

13.4.5. Reflections on Computational Tools and Environments

The ideas that we present in this section include a considerable part of the theoretical framework of the technology and of the research project. We list references to relevant works, articles, and theses derived from the project at the end of the chapter.
Working with the virtual versions of mathematical objects promotes the constructive activity of students. Indeed, these virtual versions produce the sensation of material existence, given the possibility of changing them where they exist, that is, on the screen. Students’ growing familiarization with computational tools allows these tools to be transformed into mathematical instruments (Guin & Trouche, 1999; Rabardel, 1995) in the sense that computational resources are gradually incorporated into the student’s activity. We suggest, then, that exploring with computational tools eventually allows students to realize how the mediational role of these tools helps them reorganize their problem-solving strategies. For example, when secondary school students are asked to explore the relationships between the inscribed angle in an arc and the corresponding central angle, we see two behaviors in the classroom: students remain immobilized by the question (we think this is because they are not able to mobilize their expressive resources) or, when they have computational resources at their disposal (for example, calculator TI-92), they are led to draw up comparative tables between angles and to eventually realize that the central angle is “nearly double” the inscribed angle in the same arc. The students’ strategy, taking the inscribed angle from the central angle is possible thanks to the expressive power the students acquire through the computational tools. In the absence of these, as we have already mentioned, it is not feasible for students to carry out the numerical comparison between the angles and to establish a conjecture, nor are they capable of producing a formulation associated with their explorations and express it in the language of the computational medium in which they are working. The computing environment is an abstraction domain (Noss & Hoyles, 1996), which can be understood as a scenario in which students can make it possible for their informal ideas to begin coordinating with their more formalized ideas on a subject. An abstraction domain supplies the tools so that exploration may be linked to formalization. In the example of dynamic geometry, we can put it this way: The exploration of drawings and of their properties gives rise to the recognition of a system of geometric relationships, which in the final analysis constitute the “geometric object.” This abstract object that rises out of such exploration is still “linked” to the environment: The student can talk of its general properties but use the language, the means of expression, supplied by the environment.

One of the aims of research in this field is to understand how technology implementation should be conducted. We know that the first stage could entail working within the framework of a preestablished curriculum. Successful innovations should be able to “erode” traditional curricula, however. At that point, it becomes fundamental to understand the nature of knowledge of students that emerges from their interactions with those mediating tools. Working with computational tools in school media leads us to face the work from two different angles (Berger, 1998): as amplifying tools and as cognitive reconceptualizing tools. These amplification and reconceptualization processes can be illustrated in the following way: The amplification process is similar to the function of a magnifying glass. Through this lens, we can enlarge objects visible at first sight. Magnification does not change the structure of the objects that are being observed, however, on the other hand, the reorganization process can be compared to the act of seeing through a microscope. The microscope allows us to observe what is not visible at first sight and, therefore, to enter a new plane of reality. In this way, the possibility of studying something new and of accessing new knowledge arises.

Computing environments provide a window for studying the evolving conceptions of students and teachers because they use the tools provided by that environment. Our students refer to mathematics as a set of symbolic expressions. Accordingly, knowledge of mathematics means being able to use procedures to transform a symbolic expression into another symbolic expression. Graphing tools produce a shift of attention from symbolic expressions to graphic representations. Representations are tools for understanding and mediating the way in which knowledge is constructed.
Our didactic work with computational tools led us to consider the phenomenology one can observe on the screens of calculators and computers. The screen is a space controlled from the keyboard, but that control is one of action at a distance. The desire to interact with virtual objects living on the screen provides a motivation for struggling with the complexities of a computational environment (Pimm, 1995).

Computational representations are *executable* representations, and there is an attribute of executable representations on which we want to cast light: They serve to *externalize* certain cognitive functions that formerly were executed only by people. That is the case, for instance, with the graphing of functions. During the time that passes while the graph is being drawn on the screen, the student observes the characteristics of the function that are reflected in its construction. We propose, therefore, that the student has the opportunity to transform the graph into an object of knowledge. This is similar to what the Greeks did with writing. They used the writing system not only as an external memory but also as a device to produce texts on which to reflect. As Donald (1993, p. 342) has suggested, the Greek’s critical innovation consisted of “externalizing the process of oral commentary on events.”

Explorations within an abstraction domain facilitate the understanding of the character *situated* in the propositions and the situatedness of its proofs (Moreno & Sacristan, 1998, 2000). *Situated* proofs refer to the understanding and articulation of processes within the context in which they have been explored. Let us explain: At first, students might make some observations situated within the computational environment they are exploring, and they could be able to express their observations by means of the tools and activities devised in that environment. That is the case, for instance, when the students try to invalidate (e.g., by dragging) a property of a geometric figure and they are unable to do so. That property becomes a theorem expressed via the tools and facilitated by the environment.

A situated proof is the result of a systematic exploration within an (computational) environment. It could be used to build a bridge between situated knowledge and some kind of formalization. Students purposely exploited the tools provided by the computing environment to explore mathematical relationships and to “prove” theorems (in the sense of situated proofs). Let us illustrate this point with the description of the case of a situated exploration in a classroom: In this section, we allow the description of the experience to speak for itself.

After the students acquired a certain amount of skill in the graphing of polynomial and rational functions, it was clear to them that the effect of zooming in on the graph of a function results in straightening the graph in a small interval. In other words, we can say that applying the zoom can be seen as *taking the derivative* in the graphical register of the function (Duval, 1995 & Tall, 1996, p. 310).

At this point, we considered the possibility that the didactic virtues of a cognitive conflict could promote the students’ levels of conceptualization (i.e., they could generate a powerful idea) with regard to the graphing of functions through the resources of the TI-92 calculator, for example. This manifested itself when we presented the students with the function (when graphed in the window \((-1, 1) \times (-1, 1)\))

\[
Y = (\sin(100x))/100
\]

Zooming in on any point of the graph of this function causes unexpected behavior: the new function graph reveals an oscillatory behavior that was hidden in the first graph.

There are many things that become clear through this exploration. First, this is not possible without the help of the computational resources at our disposal. Second, it allows the relationships between the graphing and the screen’s resolution to be systematized, in this case, for the TI-92 calculator. This is equal to the achievement of
a powerful idea, which includes the understanding of the screen as a representational space. Later, we have introduced the task of exploration of the function:

\[ \sum (2/3)^n \cos(9^n \pi x), n \geq 0. \]

This is known as the Weierstrass function. Historically, this function marked a milestone in the development of mathematical analysis, because it is a continuous but non-differentiable function. We do not try to give our students a formal demonstration of these mathematical characteristics, of course; rather, the idea is to use the abstraction domain supplied by the computational environment to explore whatever the student observes when graphing the polynomial approximations that correspond to the series defined by the function. We were interested in seeing what kinds of statements were put forward by the students in discussions on the process of graphing.

While observing the drawing process of the polynomial approximations of the Weierstrass function, students noticed the randomness “hidden” in such a function. They realized this characteristic of randomness because of the dynamics supported by the executable representation of the function (Lupiañez-Moreno, 2001). Once again the idea of considering the effects of the screen resolution on the graph turned out to be powerful. Through the instrumentalization of this idea, students could discover the degree of complexity of the function, perhaps only from a visual-dynamical point of view, but even this objective is worthwhile because it opens a window into
mathematical world with the potential to enhance understanding beyond the curriculum. The tool is used here as a microscope, not as a magnifying glass. There are different cognitive demands in drawing a figure by hand (Roth & McGinn, 1997) and using a computing device to accomplish that task. If we draw a circle using the border of a circular object as a guide, we obtain some valuable information on the control we have to practice with our hand. So obtained, the information can be understood to result from the mediational role of the tool (the border of the circular object we used). Drawing within a computational environment set forth a different cognitive demand from students. The nature of the mediational tools applied in each case support this assertion. There is a considerable amount of research on this topic. Recently, Chasapis (1999) discussed the mediacion of tools in the development of the concept of a circle. He suggested that human action and thought is different when students work with a compass than when they work with tracers and templates. From the first moment students access the tools as instruments to enhance their expressive power, after considerable work with the mediating help from teachers, they might enter the higher level of reconceptualization. In principle, considerable familiarity with the tools is needed to be able to produce this reconceptualization a process (Guin & Trouche, 1999).

With computer explorations, we can associate the notion of a "situated theorem," when the tools employed become visible as part of the expression. As Noss and Hoyles (1996) explained, students can generate and articulate relationships that are general to the computational environment in which they are working. This means students can develop an ability to state general propositions in the language of the environment. We can say that these computational environments derive their educational power from their ability to manipulate and externalize abstract ideas.

Now we will describe and explain some key aspects of the communal and cultural aspects of the projects described in this chapter.

13.4.6. Final Remarks on the Projects

We have described two different projects, developed within the Mexican educational system. We have also described the implementation of technology that, in the near future, might impact school practices and that can lead to new educational developments. Finally, we have tried to exhibit the type of mathematical understanding achieved by students during the implementation of these new approaches, for instance, those mediated by algebraic calculators and computers. Let us recall our description of the idea of access to powerful mathematics through school culture. It mainly
has to do with providing students with the opportunity of experiencing the construction of mathematical knowledge at school according to their level of development; with developing students' creativity through exploration and discussion of the different approaches that emerge from discussing questions posed within the classroom; and with developing students' individual computing techniques or procedures.

It must be emphasized that these projects have been conceived to respond to the real educational problems that are seen in our particular society.

We believe that in the decades ahead, these problems and the whole sociocultural environment from which the result will continue in our country as in Latin American countries in general. In particular, we will continue to face the need to take care of considerable populations with high levels of dispersion, and, at the same time, we will have to respond to the problems created by the addition of new technologies to our school systems.

The characteristics of sociocultural and economic development, which we have described in this chapter, are widely shared among Latin America. It is from this perspective that we see great potential in the educational proposals, found in our projects. These are viable projects, that go beyond the particular conditions of our country. Of course, the incorporation of information technology into school systems must be gradual, adopting a systematic approach. By this we mean that it is not just a matter of installing equipment in the absence of an educational and social project, which may lend importance to the social acceptance of these technologies. Broad social support is indispensable, and may better the quality of education and generate conditions in which new conceptual frameworks (powerful ideas) may spread to the largest possible number of schools within a country.

We want now to consider issues related to the development of the technology project from the viewpoint of the community and school culture. Researchers are aware that educational software is not culturally neutral (Crawford, 1990). For instance, the design of educational software incorporates the values and priorities of the designer. The designer's sociocultural environment will play a role—which could be an implicit role—while producing a piece of educational software. This is closely related to usability issues that we have discussed in a previous section of this chapter, and this issue is central to our project because we are using several software environments that underlie a series of powerful mathematical ideas. Many of these ideas are closely bonded to the curriculum, but others are not. The latter convey an opportunity to explore future changes that might be incorporated into the already-mentioned curriculum. Computational environments enhance students access to powerful ideas. The feasibility of dealing with general mathematical ideas within a computational environment highlights an important feature of these tools and environments: the access to systematization, a true powerful idea.

The technology project has demanded a global and local level of assessment. The global level focuses on understanding the educational system as a complex one: the interactions of students, teachers, parents, and administrators all within an educational environment. The goal of this level of assessment is to regulate the educational processes taking place at school. This includes taking care of teachers' evolving conceptions and administrators' and parents' new attitudes toward technology. On the other hand, the local level concentrates mainly on case studies. The latter is sought to provide useful feedback for improving dissemination and implementation, as well as to produce auditable trails of documentation that can reveal the nature of achievements.

Data from the local level of assessment such as filmed interviews with students were used to analyze the evolution of skills and specific knowledge according to the mathematics curriculum. Tasks were designed with a model of collaborative work in the classroom in mind. These tasks were implemented according to evolving lines in
the different curriculum contents—for instance, from intuitive to exploratory dynamic geometry.

Different pieces of software (Excel, Cabri, SimCalc) have been used at different sites, and the calculator is being used at every site. Pupils collaborate in pairs and small groups of three when working in front of the computer. When using the calculator, they work individually but also in groups. Teachers have noted that when students share a calculator, they often seemed to have an advantage. In addition, when two students each have a calculator but work together, they generate a variety of approaches and also discuss their work. The teacher’s role consisted of (a) giving support to students as they worked out the activities described in the worksheets and (b) organizing collective discussions to enhance individual experiences and problem-solving abilities. In addition to being a mediator during the classroom activities, the teacher is also a mediator between students and the tools as the students appropriate these tools.

The focus of this work is to cast light on the role of computing tools as shapers of school mathematical culture. The results discussed here provide evidence of the impact of learning environments on the ways in which children express their mathematical thinking. This is in part due to the close interaction occurring between the students and the tools. For instance, while working with the calculator, students can enter a formula and observe results of the calculations that are carried out with that formula. The student becomes aware of a broad generality expressed by the formula instead of looking only at the symbolic manipulation. In this fashion, he or she is introduced to a powerful mathematical idea. Many researchers participating in this project have observed this trend during the development and implementation of diverse activities.

We can add some remarks from the global assessment perspective:

1. Parents value technology because it brings better career opportunities to their children.
2. Teachers point out that technology helps build a new learning milieu within the classroom in which new strategies for problem solving and new ways of introducing teaching materials can emerge.

Also from a global perspective, the project tries to answer questions such as the following:

1. What new insights are productive teachers developing?
2. Are the teachers’ and parents’ expectations evolving together with the project?
3. Is the evolution of values manifested accordingly to regional cultures?

Teachers clearly do not want their involvement in the project “to be determined by the whims of elected political representatives”; they want to ensure a continuing participation in it.

When the teachers were asked about their perceptions of the quality of students’ learning, they said that they were pleased that their students were more interested in mathematics. Students were learning to reason and had become more sensitive to the introduction of mathematical ideas before they dealt with them in the normal classroom.

Teachers play a central role in helping students assimilate what they know. As professor Lesh has told us, teachers seem very comfortable with technology now and seem to be more worried about other issues within the project, such as student assessment and student commitment. Now is the time to provide teachers with the tools to consider and promote new ways of learning, not only as an internal process but also as a social event.
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